

On the normalized arithmetic Hilbert function

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Abstract

Let $X \subset \mathbb{P}_{\overline{\mathbb{Q}}}^N$ be a subvariety of dimension n , and $\mathcal{H}_{\text{norm}}(X; \cdot)$ the normalized arithmetic Hilbert function of X introduced by Philippon and Sombra. We show that this function admits the following asymptotic expansion

$$\mathcal{H}_{\text{norm}}(X; D) = \frac{\widehat{h}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad \forall D \gg 1,$$

where $\widehat{h}(X)$ is the normalized height of X . This gives a positive answer to a question raised by Philippon and Sombra.

Keywords: Arithmetic Hilbert function, Height.

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1 Introduction

In [5], Philippon and Sombra introduce an arithmetic Hilbert function defined for any subvariety in \mathbb{P}^N , the projective space of dimension N over $\overline{\mathbb{Q}}$. This function measures the binary complexity of the subvariety. In the case of toric subvarieties, a result of Philippon and Sombra shows that the asymptotic behaviour of the associated normalized arithmetic Hilbert function is related to the normalized height of the subvariety considered, see [5, Proposition 0.4]. This result is an important step toward the proof of the main theorem of [5], that is an explicit formula for the normalized height of projective translated toric varieties, see [5, Théorème 0.1].

In [5, Question 2.2], the authors ask if the normalized arithmetic Hilbert function admits an asymptotic expansion similar to the toric case. More precisely, given X a subvariety of dimension n in \mathbb{P}^N the projective space of dimension N over $\overline{\mathbb{Q}}$, can we find a real $c(X) \geq 0$ such that

$$\mathcal{H}_{\text{norm}}(X; D) = \frac{c(X)}{(n+1)!} D^{n+1} + o(D^{n+1})?$$

If this is the case, do we have $c(X) = \widehat{h}(X)$? where $\widehat{h}(X)$ is the normalized height of X .

In this article, we give an affirmative answer to this question. We prove the following theorem

Theorem 1.1. *[Theorem (2.5)] Let $X \subset \mathbb{P}^N$ be a subvariety of dimension n in \mathbb{P}^N . Then the normalized arithmetic Hilbert function associated to X admits the following asymptotic expansion*

$$\mathcal{H}_{\text{norm}}(X; D) = \frac{\widehat{h}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}), \quad \forall D \gg 1.$$

The notion of normalized height plays an important role in the diophantine approximation on tori, in particular in Bogomolov's and generalized Lehmer's problems, see [3], [2]. A result of Zhang shows that a subvariety X with a vanishing normalized height is necessarily a union of toric subvarieties, see [8].

Gillet and Soulé proved an arithmetic Hilbert-Samuel formula as a consequence of the arithmetic Riemann-Roch theorem, see [4]. Roughly speaking, this formula describes the asymptotic behaviour of the arithmetic degree of a hermitian module defined by the global sections of the tensorial power of a positive hermitian line bundle on an arithmetic variety. Moreover, the leading term is given by the arithmetic degree of the hermitian line bundle. Later Abbès and Bouche gave a new proof for this result without using the arithmetic Riemann-Roch theorem, see [1]. Randriambololona extends the result Gillet and Soulé to the case of coherent sheaf provided as a subquotient of a metrized vector bundle on an arithmetic variety, see [7].

1.1 Notations

Let \mathbb{Q} be the field of rational numbers, \mathbb{Z} the ring of integers, K a number field and \mathcal{O}_K its ring of integers. For N and D two integers in \mathbb{N} we set $\mathbb{N}_D^{N+1} := \{a \in \mathbb{N}^{N+1} \mid a_0 + \dots + a_N = D\}$. $\mathbb{C}[x_0, \dots, x_N]_D$ (resp. $K[x_0, \dots, x_N]_D$) denotes the complex vector space (resp. K -vector space) of homogeneous polynomials of degree D in $\mathbb{C}[x_0, \dots, x_N]$ (resp. in $K[x_0, \dots, x_N]$).

For any prime number p we denote by $|\cdot|_p$ the p -adic absolute value on \mathbb{Q} such that $|p|_p = p^{-1}$ and by $|\cdot|_\infty$ or simply $|\cdot|$ the standard absolute value. Let $M_{\mathbb{Q}}$ be the set of these absolute values. We denote by M_K the set of absolute values of K extending the absolute values of $M_{\mathbb{Q}}$, and by M_K^∞ the subset in M_K of archimedean absolute values.

We denote by \mathbb{P}^N the projective space over $\overline{\mathbb{Q}}$ of dimension N . A variety is assumed reduced and irreducible.

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2 The proof of Theorem (1.1)

We keep the same notations as in [5]. Let ω be the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$. For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we denote by h_k the hermitian metric on $\mathcal{O}(1)$ given as follows

$$h_k(\cdot, \cdot) = \frac{|\cdot|^2}{(|x_0|^{2k} + \dots + |x_N|^{2k})^{\frac{1}{2k}}}, \forall k \in \mathbb{N}_{\geq 1} \quad \text{and} \quad h_\infty(\cdot, \cdot) = \frac{|\cdot|^2}{\max(|x_0|, \dots, |x_N|)^2},$$

and we let $\overline{\mathcal{O}(1)}_k := (\mathcal{O}(1), h_k)$ and $\omega_k := c_1(\mathcal{O}(1), h_k)$ for any $k \in \mathbb{N} \cup \{\infty\}$. Note that $\omega_k = \frac{1}{k}[k]^*\omega$, where $[k] : \mathbb{P}^N(\mathbb{C}) \rightarrow \mathbb{P}^N(\mathbb{C})$, $[x_0 : \dots : x_N] \mapsto [x_0^k : \dots : x_N^k]$. Observe that the sequence $(\omega_k)_{k \in \mathbb{N}_{\geq 1}}$ converges weakly to the current ω_∞ . We consider the following normalized volume form

$$\Omega_k := \omega_k^{\wedge N} \quad \forall k \in \mathbb{N}_{\geq 1} \cup \{\infty\}.$$

For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, the metric of $\overline{\mathcal{O}(1)}_k$ and Ω_k define a scalar product $\mathbb{C}[x_0, \dots, x_N]_D$ denoted by $\langle \cdot, \cdot \rangle_k$ given as follows

$$\langle f, g \rangle_k := \int_{\mathbb{P}^N(\mathbb{C})} h_k^{\otimes D}(f, g) \Omega_k, \quad (1)$$

for any $f = \sum_a f_a x^a$, $g = \sum_a g_a x^a$ in $\mathbb{C}[x_0, \dots, x_N]_D$ with $f_a, g_a \in \mathbb{C}$. We denote by $\|\cdot\|_k$ the associated norm for any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Note that, $\langle f, g \rangle_\infty = \sum_a f_a \overline{g_a}$ and $\|x^a\|_\infty = 1$ for any $a \in \mathbb{N}_D^{N+1}$ and $D \in \mathbb{N}$.

Let $X \subset \mathbb{P}^N$ be a subvariety defined over a number field K . Let $v \in M_K^\infty$ and $\sigma_v : K \rightarrow \mathbb{C}$ the corresponding embedding. For any $p_1, \dots, p_l \in K[x_0, \dots, x_N]_D$ we set

$$\|p_1 \wedge \dots \wedge p_l\|_{k,v} := \|\sigma_v(p_1) \wedge \dots \wedge \sigma_v(p_l)\|_k \quad \forall k \in \mathbb{N} \cup \{\infty\}.$$

Let $\mathcal{O}(D) := \mathcal{O}(1)^{\otimes D}$. We set $M := \Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)$ the \mathcal{O}_K -module of global sections of $\mathcal{O}(D)|_\Sigma$, where Σ is the Zariski closure of X in $\mathbb{P}_{\mathcal{O}_K}^N$. For any $v \in M_K^\infty$, we set $\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)_{\sigma_v} := \Gamma(\Sigma, \mathcal{O}(D)|_\Sigma) \otimes_{\sigma_v} \mathbb{C}$. We consider the following restriction map

$$\pi : \Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))_{\sigma_v} \rightarrow \Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)_{\sigma_v} \rightarrow 0.$$

The space $\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))_{\sigma_v}$ is identified canonically to $K_\sigma[x_0, \dots, x_N]_D$. For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, this space can be endowed by the scalar product induced by Ω_k and h_k , denoted by $\langle \cdot, \cdot \rangle_{k,v}$:

$$\langle f, g \rangle_{k,v} = \langle \sigma_v(f), \sigma_v(g) \rangle_k,$$

for any $f, g \in \Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))_{\sigma_v}$. Since $\mathcal{O}(1)$ is ample, then there exists $D_0 \in \mathbb{N}$ such that for any $D \geq D_0$, the restriction map is surjective. Let $D \geq D_0$, for any $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\|\cdot\|_{k,v,\text{quot}}$ the quotient norm induced by π and $\|\cdot\|_{k,v}$. Following [5, p.348], we endow $\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)_{\sigma_v}$ with $\|\cdot\|_{k,v,\text{quot}}$, for any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. By this construction, M can be equipped with a structure of a hermitian \mathcal{O}_K -module, denoted by \overline{M}_k . If $f_1, \dots, f_s \in M$, is a K -basis for $M \otimes_{\mathcal{O}_K} K$, then

$$\widehat{\deg}(\overline{M}_k) = \widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)_k}) := \frac{1}{[K:\mathbb{Q}]} \left(\log \text{Card} \left(\bigwedge^s M / (f_1 \wedge \dots \wedge f_s) \right) - \sum_{v: K \rightarrow \mathbb{C}} \log \|f_1 \wedge \dots \wedge f_s\|_{k,v} \right).$$

2.1 The normalized arithmetic Hilbert function

Let $X \subset \mathbb{P}^N$ be a subvariety defined over a number field K and $I := I(X) \subset K[x_0, \dots, x_N]$ its ideal of definition. We set

$$\mathcal{H}_{\text{geom}}(X; D) := \dim_K(K[x_0, \dots, x_N]/I)_D = \binom{D+N}{N} - \dim_K(I_D).$$

This function $\mathcal{H}_{\text{geom}}(X; \cdot)$ is known as *the classical geometric Hilbert function*. In [5], Philippon and Sombra introduce an arithmetic analogue of this function. Let $m := \mathcal{H}_{\text{geom}}(X; D)$, $l := \dim_K(I_D)$ and

$$\bigwedge^l K[x_0, \dots, x_N]_D,$$

the l -th exterior power product of $K[x_0, \dots, x_N]_D$. For $f \in \bigwedge^l K[x_0, \dots, x_N]_D$ and $v \in M_K$ we denote by $|f|_v$ the sup-norm of the coefficients of f at the place v , with respect to the standard basis of $\bigwedge^l K[x_0, \dots, x_N]_D$.

Definition 2.1. ([5, Définition 2.1]) Let p_1, \dots, p_l be a K -basis of I_D , we set

$$\mathcal{H}_{\text{norm}}(X; D) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |p_1 \wedge \dots \wedge p_l|_v.$$

By the product formula, this definition does not depend on the choice of the basis, also it is invariant by finite extensions of K . $\mathcal{H}_{\text{norm}}(X; \cdot)$ is called the normalized arithmetic Hilbert function of X .

Following Philippon and Sombra, this arithmetic Hilbert function measures, for any $D \in \mathbb{N}$, the binary complexity of the K -vector space of forms of degree D in $K[x_0, \dots, x_N]$ modulo I . As pointed out by Philippon and Sombra, when X is a toric variety, the asymptotic behaviour of its associated normalized arithmetic Hilbert function is related to $\hat{h}(X)$, the normalized height of X , see [5, Proposition 0.4]. The authors ask the following question:

Given X a subvariety in \mathbb{P}^N of dimension n , can we find a real $c(X) \geq 0$ such that

$$\mathcal{H}_{\text{norm}}(X; D) = \frac{c(X)}{(n+1)!} D^{n+1} + o(D^{n+1})?$$

If this is the case, do we have $c(X) = \hat{h}(X)$?

We recall the following proposition, which gives a dual formulation for $\mathcal{H}_{\text{norm}}$,

Proposition 2.2. Let $q_1, \dots, q_m \in K[x_0, \dots, x_N]_D^\vee$ be a K -basis of $\text{Ann}(I_D)$, then

$$\mathcal{H}_{\text{norm}}(X; D) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |q_1 \wedge \dots \wedge q_m|_v.$$

Proof. See [5, Proposition 2.3]. □

For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we consider the following arithmetic function,

$$\begin{aligned} \mathcal{H}_{\text{arith}}(X; D, k) &:= \sum_{v \in M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|p_1 \wedge \dots \wedge p_l\|_{k,v} \\ &+ \sum_{v \in M_K \setminus M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |p_1 \wedge \dots \wedge p_l|_v + \frac{1}{2} \log(\gamma(N, D, k)), \end{aligned} \quad (2)$$

where p_1, \dots, p_l is a K -basis of I_D and

$$\gamma(N; D, k) := \prod_{a \in \mathbb{N}_D^{N+1}} \langle a, a \rangle_k^{-1}. \quad (3)$$

Notice that for $k = 1$, $\mathcal{H}_{\text{arith}}(X; \cdot, 1)$ corresponds, up to a constant, to the arithmetic function $\mathcal{H}_{\text{arith}}(X; \cdot)$ considered in [5, p. 346].

Similarly to $\mathcal{H}_{\text{norm}}$, the function $\mathcal{H}_{\text{arith}}$ admits a dual formulation. The scalar product $\langle \cdot, \cdot \rangle_k$ induces the following linear isomorphism

$$\eta_k : \mathbb{C}[x_0, \dots, x_N] \rightarrow \mathbb{C}[x_0, \dots, x_N]^\vee, \quad f \mapsto \langle \cdot, f \rangle_k.$$

Thus $\mathbb{C}[x_0, \dots, x_N]^\vee$ can be endowed with the dual scalar product, given as follows

$$\langle \eta_k(f), \eta_k(g) \rangle_k := \langle f, g \rangle_k, \quad \forall f, g \in \mathbb{C}[x_0, \dots, x_N]_D.$$

We can check easily that, for any $k \in \mathbb{N} \cup \{\infty\}$ we have $\|\theta\|'_k := \sup_{g \in \mathbb{C}[x_0, \dots, x_N] \setminus \{0\}} \frac{|\theta(g)|}{\|g\|_k} = \|f\|_k$ where $f \in \mathbb{C}[x_0, \dots, x_N]$ is such that $\theta = \eta_k(f)$. Then, $\|\theta\|'^2_k = \langle \theta, \theta \rangle_k$ for any $\theta \in \mathbb{C}[x_0, \dots, x_N]^\vee$. It follows that,

$$\langle \theta, \zeta \rangle_k = \sum_b \langle x^b, x^b \rangle_k^{-1} \theta_b \bar{\zeta}_b. \quad (4)$$

This product extends to $\wedge^m(\mathbb{C}[x_0, \dots, x_N]^\vee_D)$ as follows

$$\langle \theta_1 \wedge \dots \wedge \theta_m, \zeta_1 \wedge \dots \wedge \zeta_m \rangle_k := \det(\langle \theta_i, \zeta_j \rangle_k)_{1 \leq i, j \leq m}.$$

Proposition 2.3. *Let $q_1, \dots, q_m \in K[x_0, \dots, x_N]^\vee_D$ be a K -basis of $\text{Ann}(I_D)$, then*

$$\begin{aligned} \mathcal{H}_{\text{arith}}(X; D, k) &= \sum_{v \in M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|q_1 \wedge \dots \wedge q_m\|_{k,v}^\vee \\ &+ \sum_{v \in M_K \setminus M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |q_1 \wedge \dots \wedge q_m|_v. \end{aligned}$$

Proof. The proof is similar to [5, Proposition 2.5]. □

Lemma 2.4. *There exists D_1 such that for any $D \geq D_1$ and any $k \in \mathbb{N}$, we have*

$$\mathcal{H}_{\text{arith}}(X; D, k) = \widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)}_k) - \frac{1}{2} \mathcal{H}_{\text{geom}}(X; D) \log \binom{D+N}{N}.$$

Proof. The proof is similar to [5, lemme 2.6]. Let \mathcal{I} be the ideal sheaf of Σ and $\Gamma(\mathbb{P}_{\mathcal{O}_K}^n, \mathcal{I}\mathcal{O}(D))$ the \mathcal{O}_K -module of global sections of $\mathcal{I}\mathcal{O}(D)$, endowed with the scalar products induced by the scalar product $\langle \cdot, \cdot \rangle_k$. We claim that there exists D_1 an integer which does not depend on k such that for any $D \geq D_1$, we have

$$\widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)}_k) = \widehat{\deg}(\overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))}_k) - \widehat{\deg}(\overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{I}\mathcal{O}(D))}_k).$$

Indeed, we can find $D_1 \in \mathbb{N}$ such that $\forall D \geq D_1$, the following sequence is exact

$$0 \rightarrow \Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{I}\mathcal{O}(D)|_\Sigma) \rightarrow \Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D)) \rightarrow \Gamma(\Sigma, \mathcal{O}(D)|_\Sigma) \rightarrow 0,$$

and then by [6, lemme 2.3.6], the following sequence of hermitian \mathcal{O}_K -modules is exact

$$0 \rightarrow \overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{I}\mathcal{O}(D)|_\Sigma)}_k \rightarrow \overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))}_k \rightarrow \overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)}_k \rightarrow 0,$$

where the metrics of $\overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{I}\mathcal{O}(D)|_\Sigma)}_k$ and $\overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)}_k$ are induced by the metric of $\overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))}_k$.

We have

$$\widehat{\deg}(\overline{\Gamma(\mathbb{P}_{\mathcal{O}_K}^N, \mathcal{O}(D))}_k) = \frac{1}{2} \log(\gamma(N; D, k)) + \frac{1}{2} \binom{D+N}{N} \log \binom{N+D}{N}. \quad (5)$$

As in the proof of [5, Lemme 2.6], and keeping the same notations we have,

$$\begin{aligned} \widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)}_k) &= \frac{1}{2} \log(\gamma(N; D, k)) + \frac{1}{2} \mathcal{H}_{\text{geom}}(X; D) \log \binom{N+D}{N} \\ &+ \sum_{v \in M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \|p_1 \wedge \dots \wedge p_l\|_{k,v}^\vee - \frac{1}{[K : \mathbb{Q}]} \log \text{Card} \left(\bigwedge^l (I_{\mathcal{O}_K}) / (p_1 \wedge \dots \wedge p_l) \right). \end{aligned} \quad (6)$$

The last term in (6) does not depend on the metric. It is computed in [5, p. 349]; we have

$$\frac{1}{[K:\mathbb{Q}]} \log \text{Card} \left(\bigwedge^l (I_{\mathcal{O}_K}) / (p_1 \wedge \cdots \wedge p_l) \right) = - \sum_{v \in M_K \setminus M_K^\infty} \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]} \log |p_1 \wedge \cdots \wedge p_l|_v.$$

This ends the proof of the lemma. \square

By [7, Théorème A], we have

$$\widehat{\deg}(\overline{\Gamma(\Sigma, \mathcal{O}(D)|_\Sigma)}_k) = \frac{h_{\overline{\mathcal{O}(1)}_k}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad \forall D \gg 1, \quad (7)$$

where $h_{\overline{\mathcal{O}(1)}_k}(X)$ denotes the height of the Zariski closure of X in $\mathbb{P}_{\mathcal{O}_K}^N$ with respect to $\overline{\mathcal{O}(1)}_k$. Since $\frac{1}{2} \mathcal{H}_{\text{geom}}(X; D) \log \binom{D+N}{N} = o(D^{n+1})$ for $D \gg 1$. Then, by Lemma (2.4), we get

$$\mathcal{H}_{\text{arith}}(X; D, k) = \frac{h_{\overline{\mathcal{O}(1)}_k}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad \forall D \gg 1. \quad (8)$$

Let $q_1, \dots, q_m \in K[x_0, \dots, x_N]^\vee$ be a K -basis of $\text{Ann}(I_D)$. For any finite subset M in \mathbb{N}_D^{N+1} of cardinal m , we set $q_M := (q_{jb})_{1 \leq j \leq m, b \in M} \in K^{m \times m}$ where the q_{jb} are such that $q_j = \sum_{b \in \mathbb{N}_D^{N+1}} q_{jb}(x^b)^\vee$. For any $v \in M_K^\infty$, we have

$$\begin{aligned} |q_1 \wedge \cdots \wedge q_m|_v &= \max\{|\det(q_M)|_v : M \subset \mathbb{N}_D^{N+1}, \text{Card}(M) = m\} \\ &\leq \left(\sum_{M; \text{Card}(M)=m} \left(\prod_{b \in M} \langle b, b \rangle_{v,k}^{-1} \right) |\det(q_M)|_v^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (9)$$

(We use the following inequality: $\langle x^a, x^a \rangle_k = \int_{\mathbb{P}^N(\mathbb{C})} h_{\overline{\mathcal{O}(D)}_k}(x^a, x^a) \Omega_k \leq 1$ for any $a \in \mathbb{N}_D^{N+1}$, which follows from $h_{\overline{\mathcal{O}(D)}_k}(x^a, x^a) \leq h_{\overline{\mathcal{O}(D)}_\infty}(x^a, x^a) \leq 1$ on $\mathbb{P}^N(\mathbb{C})$, and the fact that Ω_k is positive on $\mathbb{P}^n(\mathbb{C})$ and $\int_{\mathbb{P}^N(\mathbb{C})} \Omega_k = 1$).

Then,

$$|q_1 \wedge \cdots \wedge q_m|_v \leq \|q_1 \wedge \cdots \wedge q_m\|_{k,v}^\vee \quad \forall k \in \mathbb{N}. \quad (10)$$

By Propositions (2.2) and (2.3) we get,

$$\mathcal{H}_{\text{norm}}(X; D) \leq \mathcal{H}_{\text{arith}}(X; D, k) \quad \forall k \in \mathbb{N}. \quad (11)$$

By (8), the previous inequality gives

$$\limsup_{D \rightarrow \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) \leq h_{\overline{\mathcal{O}(1)}_k}(X) \quad \forall k \in \mathbb{N}. \quad (12)$$

We know that $(h_k)_{k \in \mathbb{N}}$ converges uniformly to h_∞ on $\mathbb{P}^N(\mathbb{C})$. Let $0 < \varepsilon < 1$, which will be fixed in the sequel, then there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, we have

$$(1 - \varepsilon)^{2D} \leq \frac{(\max(|x_0|_v, \dots, |x_N|_v))^{2D}}{(|x_0|_v^{2k} + \cdots + |x_N|_v^{2k})^{\frac{D}{k}}} \leq (1 + \varepsilon)^{2D} \quad \forall x \in \mathbb{P}^N(\mathbb{C}), \forall D \in \mathbb{N}.$$

Thus, for any $k \geq k_0$, $D \in \mathbb{N}_{\geq 1}$ and $a \in \mathbb{N}_D^{N+1}$ we get

$$\langle x^a, x^a \rangle_k \geq (1 - \varepsilon)^{2D} \int_{\mathbb{P}^N(\mathbb{C})} h_\infty^{\otimes D}(x^a, x^a) \omega_k^N. \quad (13)$$

We have

$$\begin{aligned}
\int_{\mathbb{P}^N(\mathbb{C})} h_{\infty}^{\otimes D}(x^a, x^a) \omega_k^N &= \int_{\mathbb{C}^N} \frac{|z^{2a}|}{\max(1, |z_1|, \dots, |z_N|)^{2D}} \frac{k^N \prod_{i=1}^N |z_i|^{2(k-1)} \prod_{i=1}^N dz_i \wedge d\bar{z}_i}{(1 + \sum_{i=1}^N |z_i^{2k}|)^{N+1}} \\
&= 2^N \int_{(\mathbb{R}^+)^N} \frac{k^N r^{a+k-1}}{\max(1, r_1, \dots, r_N)^{2D}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i^k)^{N+1}} \\
&= 2^N \int_{(\mathbb{R}^+)^N} \frac{r^{\frac{a}{k}}}{\max_i(1, r_1, \dots, r_N)^{\frac{D}{k}}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i^k)^{N+1}} \\
&= 2^N \sum_{j=0}^N \int_{E_j} \frac{r^{\frac{a}{k}}}{\max_i(1, r_1, \dots, r_N)^{\frac{D}{k}}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}},
\end{aligned}$$

where $E_j := \{x \in (\mathbb{R}^+)^N \mid x_j \geq 1, x_l \leq x_j \text{ for } l = 1, \dots, N\}$ for $j = 1, \dots, N$ and $E := \{x \in (\mathbb{R}^+)^N \mid x_l \leq 1, \text{ for } l = 1, \dots, N\}$. Using the following application

$$(\mathbb{R}^{*+})^N \rightarrow (\mathbb{R}^{*+})^N, \quad x = (x_1, \dots, x_N) \mapsto \left(\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{1}{x_j}, \dots, \frac{x_N}{x_j}\right)$$

for $j = 1, \dots, N$, we can show that there exists $b^{(j)} = (b_1^{(j)}, \dots, b_N^{(j)}) \in \mathbb{N}^N$ such that

$$\int_{E_j} \frac{r^{\frac{a}{k}}}{\max_i(1, r_1, \dots, r_N)^{\frac{D}{k}}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} = \int_E r^{\frac{b^{(j)}}{k}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} \quad (14)$$

We set $b^{(0)} := a$. Then,

$$\int_{\mathbb{P}^N(\mathbb{C})} h_{\infty}^D(x^a, x^a) \omega_k^N = 2^N \sum_{j=0}^N \int_E r^{\frac{b^{(j)}}{k}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} \quad (15)$$

Let $0 < \delta < 1$, and set $E_{\delta} := \{x \in E \mid x_l \geq \delta \text{ for } l = 1, \dots, N\}$. From (13) and (15), we obtain

$$\langle x^a, x^a \rangle_k \geq (1 - \varepsilon)^{2D} 2^N \sum_{j=0}^N \int_{E_{\delta}} r^{\frac{b^{(j)}}{k}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} \geq (1 - \varepsilon)^{2D} 2^N (N + 1) \delta^{\frac{D}{k}} \mu_{\delta},$$

where $\mu_{\delta} := \int_{E_{\delta}} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}}$.

Thus,

$$\langle x^a, x^a \rangle_k^{-1} \leq (1 - \varepsilon)^{-2D} \delta^{-\frac{D}{k}} \mu_{\delta}^{-1} \quad \forall k \geq k_0, \forall D \in \mathbb{N}_{\geq 1}, \forall a \in \mathbb{N}_D^{N+1}. \quad (16)$$

Then, for any $k \geq k_0$ and $D \geq D_1$,

$$\begin{aligned}
\|q_1 \wedge \dots \wedge q_m\|_{k,v}^{\vee} &\leq \left(\sum_{M; \text{Card}(M)=m} \left(\prod_{b \in M} \langle b, b \rangle_{v,k}^{-1} \right) \right)^{\frac{1}{2}} |q_1 \wedge \dots \wedge q_m|_v \\
&\leq \text{Card}\{M \subset \mathbb{N}_D^{N+1} \mid \text{Card}(M) = m\}^{\frac{1}{2}} (1 - \varepsilon)^{-mD} \delta^{-m\frac{D}{k}} \mu_{\delta}^{-m} |q_1 \wedge \dots \wedge q_m|_v \quad \text{by (16)} \\
&\leq \text{Card}(\mathbb{N}_D^{N+1}) (1 - \varepsilon)^{-mD} \delta^{-m\frac{D}{k}} \mu_{\delta}^{-m} |q_1 \wedge \dots \wedge q_m|_v \\
&= \binom{N+D}{N}^{\frac{1}{2}} (1 - \varepsilon)^{-mD} \delta^{-m\frac{D}{k}} \mu_{\delta}^{-m} |q_1 \wedge \dots \wedge q_m|_v.
\end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} \mathcal{H}_{\text{arith}}(X; D, k) &\leq \mathcal{H}_{\text{norm}}(X; D) + \frac{1}{2} \log \binom{N+D}{N} - D\mathcal{H}_{\text{geom}}(X; D) \log(1 - \varepsilon) \\ &\quad - \frac{D\mathcal{H}_{\text{geom}}(X; D)}{k} \log \delta - \mathcal{H}_{\text{geom}}(X; D) \log \mu_\delta. \end{aligned} \quad (18)$$

By (8), we obtain that

$$h_{\overline{\mathcal{O}(1)}_k}(X) \leq \liminf_{D \rightarrow \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) + O(\varepsilon) + \frac{\log \delta}{k} O(1), \quad \forall k \geq k_0. \quad (19)$$

Gathering (12) and (19), we conclude that for any $0 < \varepsilon < 1$, there exists $k_0 \in \mathbb{N}$ such that

$$\limsup_{D \rightarrow \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) \leq h_{\overline{\mathcal{O}(1)}_k}(X) \leq \liminf_{D \rightarrow \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) + O(\varepsilon) + \frac{\log \delta}{k} O(1), \quad \forall k \geq k_0. \quad (20)$$

Since $\lim_{k \rightarrow \infty} h_{\overline{\mathcal{O}(1)}_k}(X) = h_{\overline{\mathcal{O}(1)}_\infty}(X)$ (see for instance [9]) and $h_{\overline{\mathcal{O}(1)}_\infty}(X) = \widehat{h}(X)$ (see [5, p. 342]) we get

$$\liminf_{D \rightarrow \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) = \limsup_{D \rightarrow \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) = \widehat{h}(X). \quad (21)$$

Thus we proved the following theorem

Theorem 2.5. *Let $X \subset \mathbb{P}^N$ be a subvariety of dimension n in \mathbb{P}^N . Then the normalized arithmetic Hilbert function associated to X admits the following asymptotic expansion*

$$\mathcal{H}_{\text{norm}}(X; D) = \frac{\widehat{h}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad D \gg 1.$$

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